Constraint Satisfaction Problems and Projective Clone Homomorphisms

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Constraint Satisfaction Problems

Let Γ be a structure. The constraint satisfaction problem of Γ, $\mathrm{CSP}(\Gamma)$, is the problem of deciding given a finite system of constraints over the variables $x_1, \dots, x_n$ if there exist elements $v_1, \dots, v_n$ in Γ that satisfy all the constraints.

Clones

A set $\mathscr{C}$ of functions of finite arity on a set X is a function clone if:

- all the projections $p_i^{(n)} \colon (x_1, \dots, x_n) \mapsto x_i$ are in $\mathscr{C}$ for $n \in \mathbb{N}$ and $1 \le i \le n$,
- $\mathscr{C}$ is closed under composition.

By endowing a function clone with the topology of pointwise convergence, one gets a topological clone.

Example

Let K_r be the complete graph on r vertices. Then $\mathrm{CSP}(K_r)$ is the r-colourability problem: the variables represent vertices of the input graph, and the constraints are "no two adjacent nodes can be mapped to the same vertex of K_r (i.e., to the same colour)".

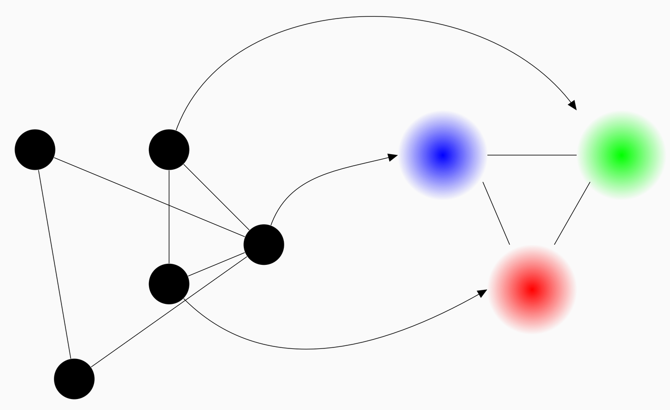

Fig. 1: $\mathrm{CSP}(K_3)$ is the 3-colourability problem.

Polymorphism Clones

Let D be a set, $R \subseteq D^n$ be an n-ary relation on D and $f \colon D^m \to D$ be an m-ary function on D. We say that f preserves R if for all tuples $(a_{11}, \dots, a_{1n}), \dots, (a_{m1}, \dots, a_{mn})$ in R, the tuple $(f(a_{11}, \dots, a_{m1}), \dots, f(a_{1n}, \dots, a_{mn}))$ is in R (see Figure 2).

$$\begin{array}{ccccc} (a_{11} & a_{12} & \dots & a_{1n}) & \in R \\ & & \vdots & & \\ (a_{m1} & a_{m2} & \dots & a_{mn}) & \in R \\ \downarrow & \downarrow & \dots & \downarrow & \\ (f(a_{11}, \dots, a_{m1}) & f(a_{12}, \dots, a_{m2}) & \dots & f(a_{1n}, \dots, a_{mn})) & \in R \end{array}$$

Fig. 2: Preservation of relations under an operation

A function f is a polymorphism of a structure Γ if f preserves all the relations of Γ.

Fact. The set of polymorphisms of a structure is a function clone, denoted by $\mathrm{Pol}(\Gamma)$. It is furthermore a closed subset of the set of all finitary functions under the topology of pointwise convergence.

Connecting Computational Complexity and the Study of Clones

There are decision problems that cannot be formulated as the CSP of a structure with a finite domain. However, we can often formulate these problems as the CSP of structures which are ω-categorical: a structure Γ is ω-categorical when every structure Δ that satisfies the same first-order properties as Γ is isomorphic to Γ. In particular, every finite structure is ω-categorical.

A clone homomorphism from a clone $\mathscr{C}$ to a clone $\mathscr{D}$ is a function ξ that maps $p_i^{(n)}$ to $p_i^{(n)}$ for all $n \in \mathbb{N}$ and $1 \le i \le n$ and such that

$$\xi(f \circ (g_1, \dots, g_n)) = \xi(f) \circ (\xi(g_1), \dots, \xi(g_n)).$$

Theorem 1 ([BP14]). *Let Γ, Δ be two ω-categorical structures. If there is a continuous clone homomorphism $\xi \colon \mathrm{Pol}(\Gamma) \to \mathrm{Pol}(\Delta)$, then $\mathrm{CSP}(\Delta)$ reduces in polynomial time to $\mathrm{CSP}(\Gamma)$.*

The clone $\mathbf{2}$ is the smallest function clone on the set $\{0, 1\}$: it only contains the projections. This clone is the polymorphism clone of every NP-hard CSP on two elements (assuming $\mathsf{P} \neq \mathsf{NP}$). A clone homomorphism from a clone $\mathscr{C}$ to $\mathbf{2}$ is called a projective clone homomorphism.

Corollary 2 (Corollary of Theorem 1). *Let Γ be an ω-categorical structure. If there is a continuous projective homomorphism $\xi \colon \mathrm{Pol}(\Gamma) \to \mathbf{2}$, then $\mathrm{CSP}(\Gamma)$ is NP-hard.*

It is conjectured that for a very general class of structures, having a continuous projective homomorphism is the only source of intractability.

Our first result confirms this conjecture for a wide class of infinite structures. Our second result uses CSP-related techniques to study the model-checking problem for the logic MMSNP.

The Complexity of First-Order Reducts of $(\mathbb{N}; 0)$

Let Γ be a structure whose domain is $\mathbb{N}$ and whose relations can all be expressed by first-order formulas in $(\mathbb{N}; 0)$. Such a structure is called a first-order reduct of $(\mathbb{N}; 0)$. Examples of problems that can be expressed as the CSP of a first-order reduct of $(\mathbb{N}; 0)$ are all the boolean CSPs, and all the equality constraint languages.

Theorem 3 ([BBM]). *Let Γ be a first-order reduct of $(\mathbb{N}; 0)$. Then $\mathrm{CSP}(\Gamma)$ is in P or NP-complete.*

Our proof works by studying the polymorphism clones of reducts of $(\mathbb{N}; 0)$ and by finding algorithms in the case where the clones do not have a continuous projective homomorphism.

A Dichotomy Theorem for a Fragment of MMSNP

A second-order existential formula ϕ is in MMSNP if it is of the form $\exists U_1, \dots, U_k \forall \overline{x}\, \phi(\overline{U}, \overline{x})$ where ϕ is a quantifier-free formula where all the symbols other than $U_1, \dots, U_k$ appear negatively, and ϕ does not contain $=$ or $\neq$. It is conjectured [FV99] that for every formula ϕ in MMSNP, the problem of deciding $A \models \phi$ given a finite structure A is in P or NP-complete (while if $\mathsf{P} \neq \mathsf{NP}$, there are decision problems with intermediate complexity), i.e., that MMSNP exhibits a complexity dichotomy.

With each MMSNP formula we can associate a coloured obstruction set $\mathcal{F}$, which is a finite set of coloured structures such that $G \models \phi$ iff there exists a colouring G^* of G such that $F \not\to G^*$ for every $F \in \mathcal{F}$.

Theorem 4 ([BM15]). *Let ϕ be a formula in MMSNP. Suppose that ϕ has a coloured obstruction set that is 2-connected and monochromatic. Then the model-checking problem for ϕ is polynomial-time solvable or NP-complete. Moreover, verifying if the model-checking problem for ϕ is in P is decidable.*

References

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