Hyperbolic polynomials, interlacing families, and the Kadison-Singer theorem

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The Kadison-Singer Problem

The *Kadison-Singer problem* was a problem in functional analysis, that came from work work on the fundations of quantum mechanics, and was recently proven by Marcus Spielman and Srivastava in [1]. Their proof makes use of the method of *interlacing families*, which is a method used to bound the maximum root of a familiy of polynomials, in conjunction with theorems and results on hyperbolic polynomials. We present some examples of applications of this method.

Interlacing families

We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces* a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1} \leq \beta_n$$

and we note it $g \ll f$. We say that polynomials $f_1, \ldots, f_k$ have a *common interlacing* if there is a polynomial g so that g interlaces f_i for each i. We also say that $\{f_i\}$ is an *interlacing family*.

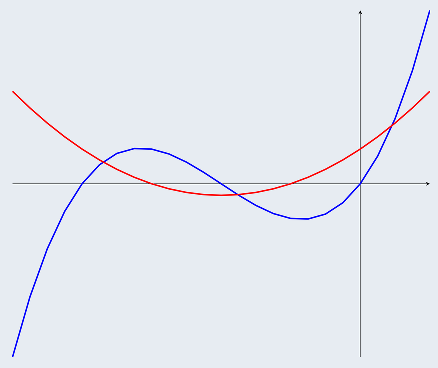

Figure 1: The polynomial in red interlaces the polynomial in blue

Ramanujan Graphs

A Ramanujan Graph is a d-regular graph which non-trivial eigenvalues lie in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. Using the method of interlacing families, Marcus Spelman and Srivastava constructed bipartite Ramanujan graphs of any degree. Their proof used Biliu and Linial's 2-lifts, introduced in [2], to create a family of characteristic polynomials $\{f_s\}$ where s is a 2-lift.

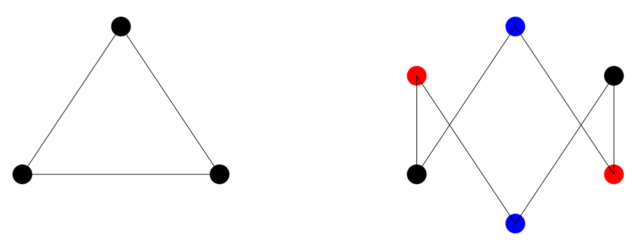

Figure 2: An example of 2-lift

Marcus Spielman and Srivastava managed to prove this family to be interlacing, and that

$$\mathbf{E} f_s = \mu_G$$

where μ_G is the matching polynomial of the graph G. Since we know the maximum root of this polynomial, they could conclude that if G is bipartite Ramanujan, then there exist a 2-lift of G that is also bipartite Ramanujan.

Our work consisted in proving a similar result for non bipartite Ramanujan graphs. We found that the problem could be reduced to finding the maximum root of

$$t \mapsto \mathtt{MAP}_{\mathbf{y}} \prod_{v \in V} \Big(t - \sum_{v \in e}(y_e - 1)\Big)^2 \Big|_{y_e = 1}$$

which happens to be very similar to the bipartite case.

Partially ordered sets

For a partially ordered set (poset), we can define its Eulerian polynomial as the polynomial counting its linear extensions by their number of descents.

$$A_{(P,\omega)} = \sum_{(r,\Pi) \in \mathcal{L}(P)} t^{des(r,\Pi)}$$

The Negger-Stanley conjecture asserts that this polynomial always has real zeros. Brändén found counter examples to this conjecture in [3], but some families of posets, like trees, verify this property, as proven by Wagner in [4].

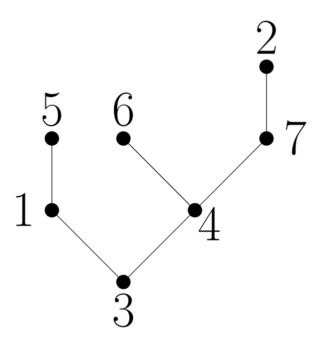

Figure 3: A Labelled Poset Tree

Using properties of the *diamond product*, we managed to prove that the family of partial Eulerian polynomials is an interlacing family for some kinds forests, and we believe that it still holds for any kind of forest.

The two-species ASEP

The asymmetric simple exclusion process (ASEP) is a model from statistical physics that describes the dynamics of interacting particles hopping left and right on a one-dimensional finite lattice with open boundaries. We are interested in a modified exclusion ASEP, with two kinds of particles: light and heavy.

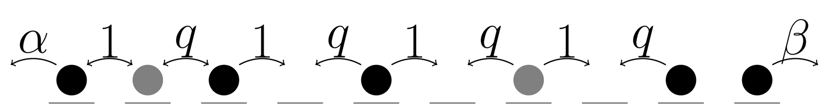

Figure 4: The parameters of the two-species ASEP

The partition function of the ASEP $P_{n,r}(x, y)$ is defined as the coefficient of z^r in $\frac{\langle w|(xD+zA+yE)^n|v\rangle}{\langle w|A^r|v\rangle}$, where n is the size of the lattice and r is the number of light particles. Let $Q_n = \sum_{k=0}^{n} z^k P_{n,k} = \frac{\langle w|(xD+zA+yE)^n|v\rangle}{\langle w|A^r|v\rangle}$. We managed to prove that the following equation when $q = 1$

$$Q_n = \frac{\partial}{\partial z}\left((z + x)(z + y) Q_{n-1}\right)$$

This recursion preserves stability, which implies that the stationary probability of it Markov Chain is strongly Raighley.

Hyperbolic polynomials

A homogeneous polynomial is a polynomial whose nonzero terms all have the same degree. A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$, and if for all $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(t\mathbf{e} - \mathbf{x})$ has only real zeros.

Spanning trees

Brändén proved in [5] a sufficient condition for graphs to have disjoint spanning trees

Let $G = (V, E)$ be a graph. If, for all $e \in E$,

$$\mathbb{P}[e] \leq \left(\frac{1}{\sqrt{k-1}} - \frac{1}{\sqrt{k}}\right)^2$$

then there exists k disjoint spanning trees in G

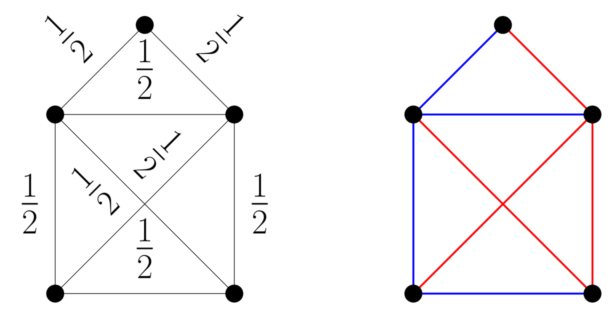

Figure 5: Two disjoint spanning trees in a graph

For each $T \in \mathcal{T}$ the set of spanning trees of G, if $a(T) = \prod_{e \in T} x_e$, then the polynomial $P = \sum_{T \in \mathcal{T}} a(T)$ is hyperbolic with respect to the all ones vector. Using the method of interlacing families to this problem, we can prove the following

Let $G = (V, E)$ be a graph, and $n = |V|$. If, for all $e \in E$,

$$\mathbb{P}[e] \leq \sqrt{\frac{2}{n(n-1)}}$$

then there exists 2 disjoint spanning trees in G

References

[1] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. Interlacing families 2: Mixed characteristic polynomials and the kadison-singer problem. *arXiv:1306.3969v4 [math.CO]*, 2014.

[2] Yonatan Bilu and Nathan Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 2006.

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[4] David G. Wagner. Enumeration of functions from posets to chains. *European Journal of Combinatorics*, 1992.

[5] Petter Brändén. Hyperbolic polynomials and the marcus-spielman-srivastava theorem. *arXiv:1412.0245v1 [math.CO]*, 2014.

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