Natural computation

Automata networks

Sylvain Sené

LABORATOIRE
D'INFORMATIQUE
FONDAMENTALE
de Marseille

ENS Cachan visits Marseille

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Plan

1. Preliminaries
2. Some known results and open questions

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2. Some known results and open questions

Definitions

A Boolean automata network (BAN) of size n is a function

$$f: \begin{array}{ccc} \mathbb{B}^n & \to & \mathbb{B}^n \\ x = (x_0, x_1, \dots, x_{n-1}) & \mapsto & f(x) = (f_0(x), f_1(x), \dots, f_{n-1}(x)) \end{array},$$

where $\forall i \in \{0, \dots, n-1\}$, $x_i \in \mathbb{B}$ is the state of automaton i, and $\mathbb{B}^n$ is the set of configurations.

The interaction graph of f is the signed digraph $G(f) : (V, E \subseteq V \times V)$ where:

- $V = \{0, \dots, n-1\}$;
- $(j,i) \in E$ is positive if $\exists x \in X = \{0,1\}^n$ s.t. $f_i(x_0, \dots, x_{j-1}, \mathbf{0}, x_{j+1}, \dots, x_{n-1}) = 0$ et $f_i(x_0, \dots, x_{j-1}, \mathbf{1}, x_{j+1}, \dots, x_{n-1}) = 1$;
- $(j,i) \in E$ is negative if $\exists x \in X = \{0,1\}^n$ s.t. $f_i(x_0, \dots, x_{j-1}, \mathbf{0}, x_{j+1}, \dots, x_{n-1}) = 1$ et $f_i(x_0, \dots, x_{j-1}, \mathbf{1}, x_{j+1}, \dots, x_{n-1}) = 0$.

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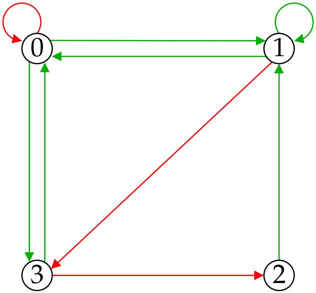
$$f \quad : \quad \{0,1\}^4 \to \{0,1\}^4$$

$$f \quad = \quad \begin{pmatrix} f_0(x) = \neg x_0 \vee x_1 \wedge x_3 \\ f_1(x) = x_0 \wedge (x_1 \vee x_2) \\ f_2(x) = \neg x_3 \\ f_3(x) = x_0 \vee \neg x_1 \end{pmatrix}$$



Automata updates

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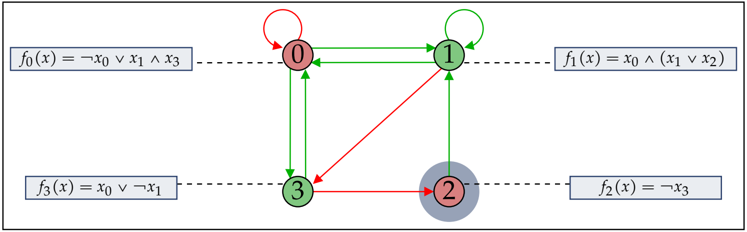

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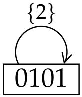

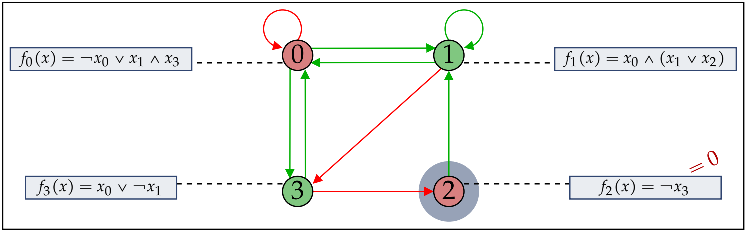

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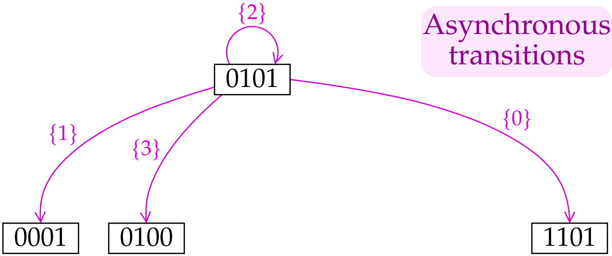
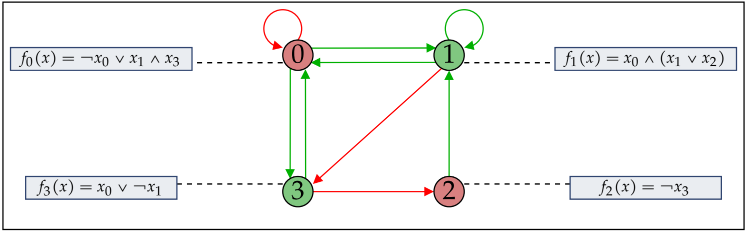


Asynchronous transitions



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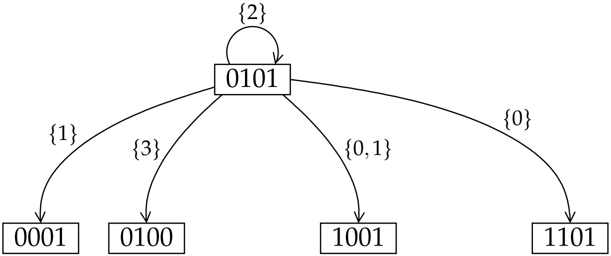

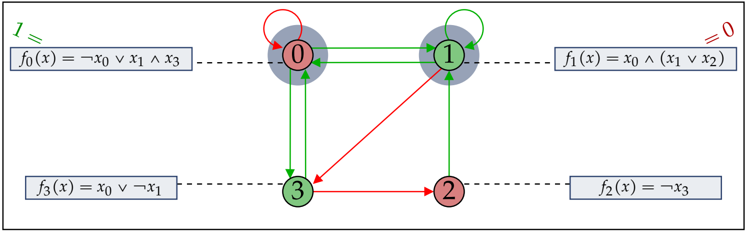

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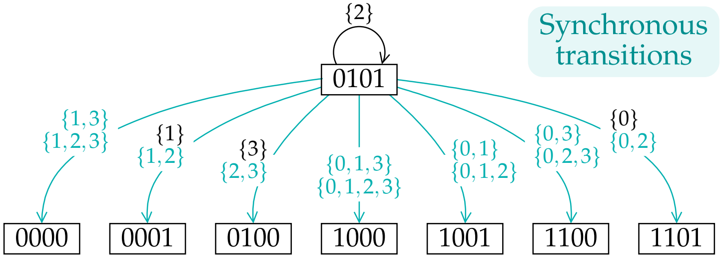
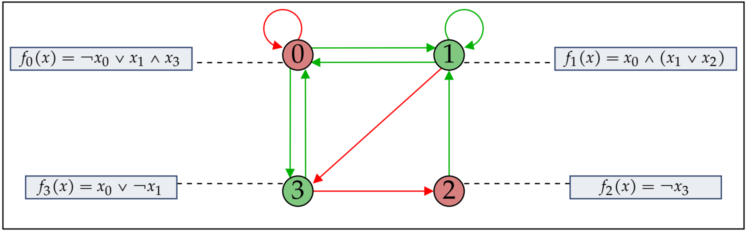


Synchronous transitions



BAN behaviour

Update modes

The update mode   the network behaviour

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The behaviour of a BAN f is described by a transition graph

$$\mathscr{G}_{\blacklozenge}(f) = (\{0,1\}^n, T \subseteq \{0,1\}^n \times (\mathscr{P}(V) \setminus \emptyset) \times \{0,1\}^n),$$

where $\blacklozenge$ represents a given "fair" update mode.

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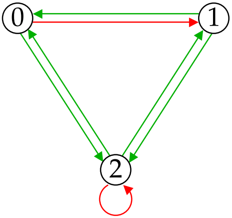

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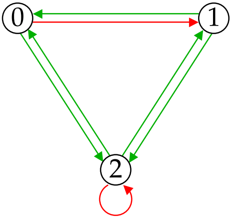

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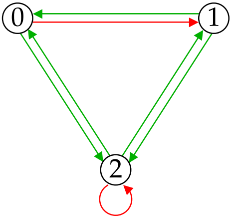

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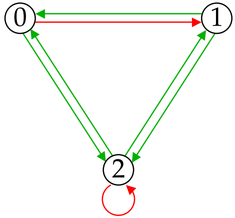

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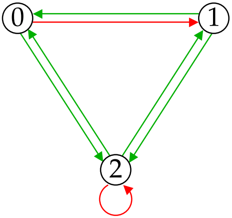

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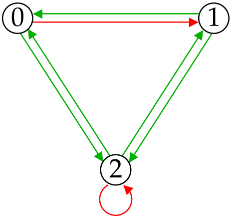
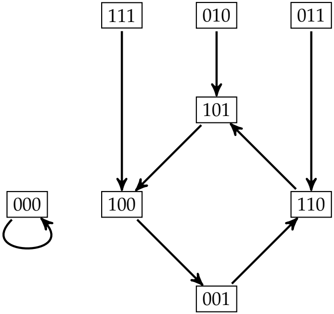


BAN behaviour

Parallel evolution

The parallel transition graph of f is the digraph $\mathscr{G}_\pi(f) = (\{0,1\}^n, T)$, with

$$T = \{(x, V, f(x)) \mid x \in \{0,1\}^n\}.$$



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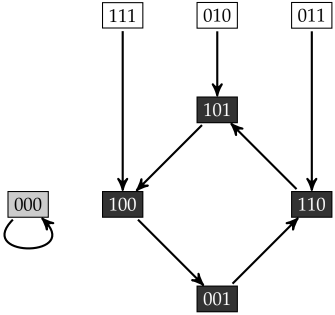


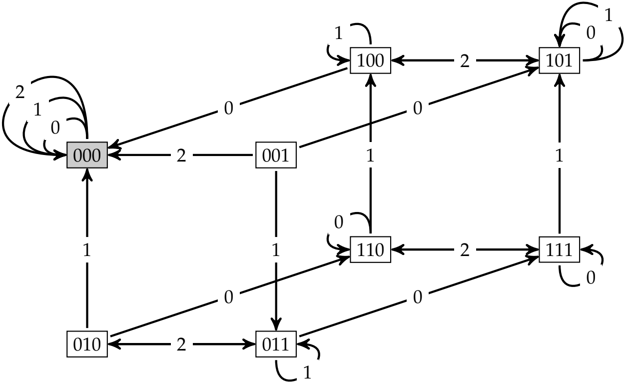
- An attractor of $(f, \blacklozenge)$ is a terminal SCC of de $\mathscr{G}_\blacklozenge(f)$.
- A fixed point (stable configuration) is a trivial attractor.
- A limit cycle (stable oscillation) is a non trivial attractor.

BAN behaviour

Asynchronous update mode

The asynchronous transition graph of f is the digraph $\mathscr{G}_\alpha(f) = (\{0,1\}^n, T)$, with:

$$T = \{(x,i,y) \mid x,y \in \{0,1\}^n \text{ et } y = (x_0, \ldots, x_{i-1}, f_i(x), x_{i+1}, \ldots, x_{n-1})\}.$$



Plan

About computability

◇ Turing universality

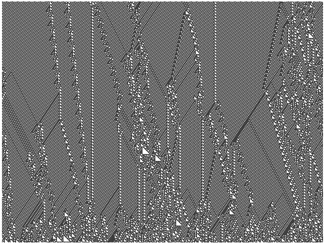
Theorem (McCulloch & Pitts, 1943)

Threshold BANs are Turing-complete.

Idea of a neat proof

→ Any Turing machine can be simulated by a cellular automaton (Smith, 1971).

→ Any cellular automaton can be simulated by a threshold BAN (Goles & Martínez, 1990).

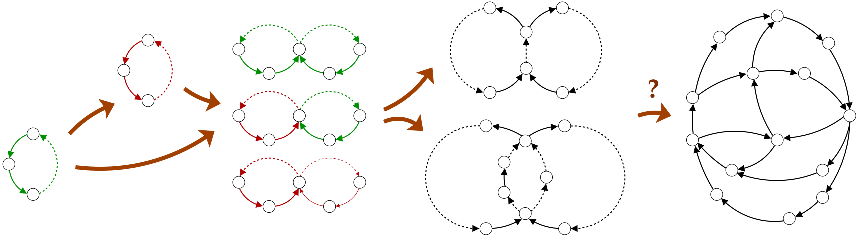


About computability

◇ Intrinsic simulation

Problem (Demongeot, Noual & S., 2012)

Cycles, double-cycles... What else?

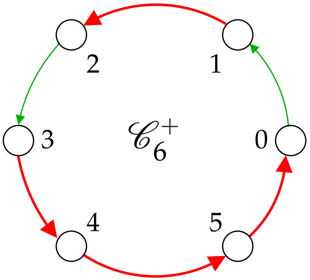


Problem (Bridoux, Guillon, Perrot, S. & Theyssier, 2017)

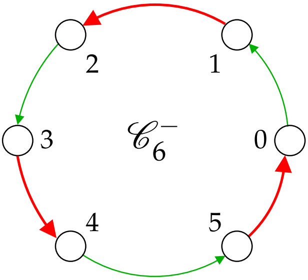
Simulating $(f, \blacklozenge_1)$ by $(g, \blacklozenge_2)$.
$\rightarrow$ Possible? How? At which cost?...

Interaction cycles

- 2 kinds of cycles: the positive and the negative ones.



an even number of negative arcs



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About counting and characterising

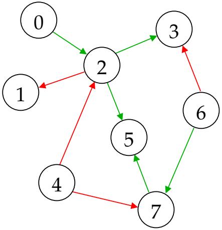
Theorem (Robert, 1986)

No cycles in $G(f) \implies \mathscr{G}_{\blacklozenge}(f)$ admits a unique attractor, a fixed point.

Idea of the proof

→ Remark that $\forall i \in V, \delta_i^- = 0$, $f_i(x) = \{0,1\}$.

→ Take a DAG and use an induction on automata depth.



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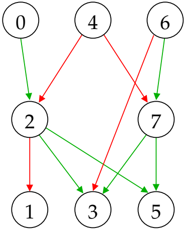

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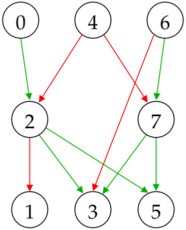

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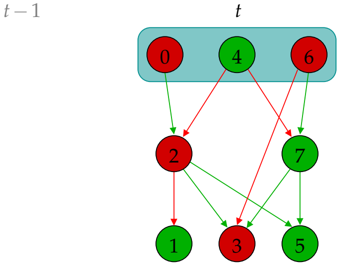

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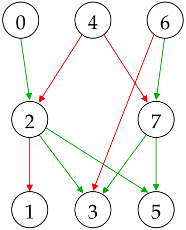

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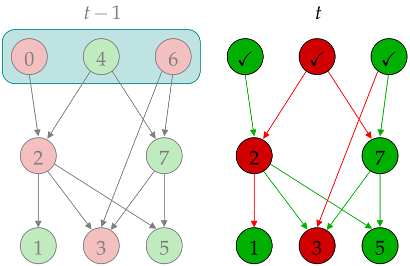

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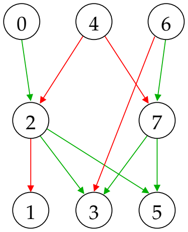
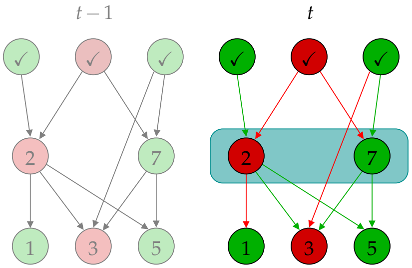


$$\begin{cases} f_0(x) = 1 \\ f_1(x) = \neg x_2 \\ f_2(x) = x_0 \wedge \neg x_4 \\ f_3(x) = x_2 \vee (\neg x_6 \wedge x_7) \\ f_4(x) = 0 \\ f_5(x) = x_2 \wedge x_7 \\ f_6(x) = 1 \\ f_7(x) = \neg x_4 \vee x_6 \end{cases}$$



About counting and characterising

Theorem (Robert, 1986)

No cycles in $G(f) \implies \mathscr{G}_{\blacklozenge}(f)$ *admits a unique attractor, a fixed point.*

Idea of the proof

→ Remark that $\forall i \in V, \delta_i^- = 0$, $f_i(x) = \{0,1\}$.

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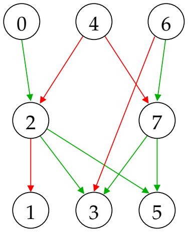

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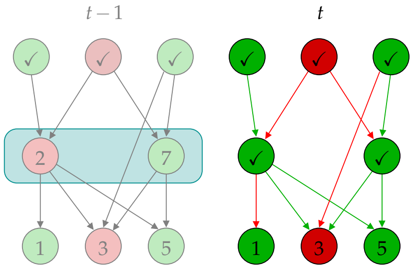

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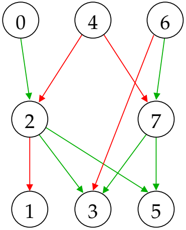

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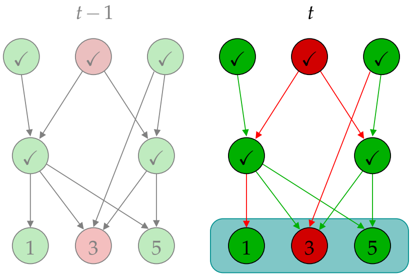

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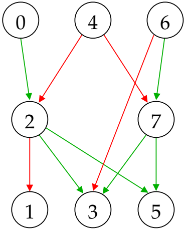

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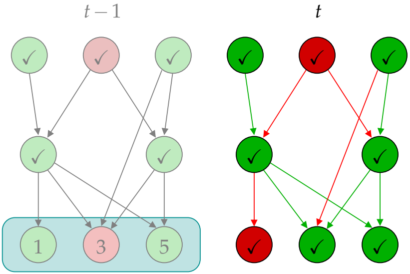

About counting and characterising

Theorem (Thomas, 1981; Richard & Comet, 2007; Noual & S., 2012)

$\mathscr{G}_{\blacklozenge}(f)$ admits several fixed points $\implies$ G contains a positive cycle.

History of this result

- → Proposed in 1981 as a conjecture.
- → Proven in 2007 for $\blacklozenge = \alpha$.
- → Generalised to any $\blacklozenge$ in 2012.

Idea of the proof

- → Admit the result of 2007.
- → Consider the elementary update mode.
- → For any BAN with no positive cycles:
  - Either it doesn't contain any cycle and Robert's theorem holds.
  - Or it does contain a negative cycle, which prevents removing all the local unstabilities.

About counting and characterising

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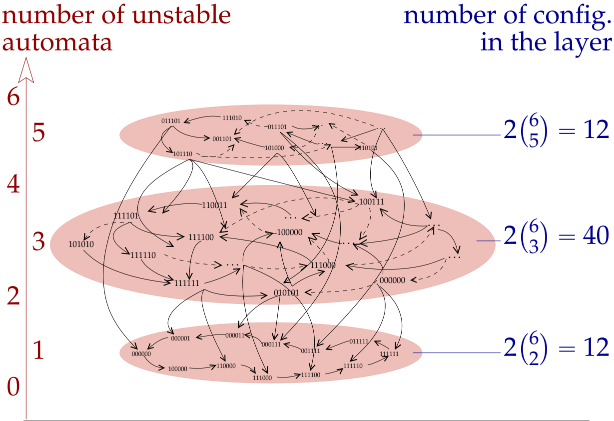
Example Elementary dynamics of a negative 6-cycle



About counting and characterising

Problem

Given an interaction graph $G = (V, E)$, determine $\phi(G) = max(\{card(FP(G,f)) \mid f : \mathbb{B}^n \to \mathbb{B}^n\})$.

→ $\tau^+(G)$: *positive feedback vertex set* of G, *i.e.* min. number of automata meeting all the positive cycles.
→ $\nu(G)$: *packing number* of G, *i.e.* mac number of disjoint cycles in G.

Theorem (Aracena, 2008; Aracena, Richard & Salinas, 2017)

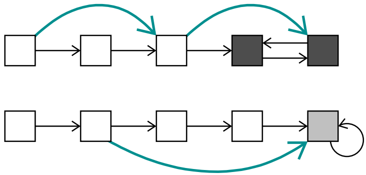
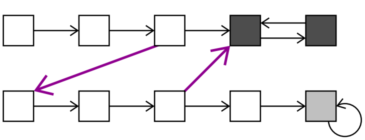
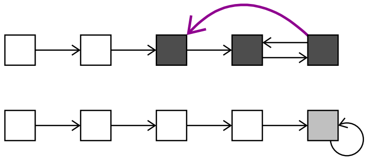
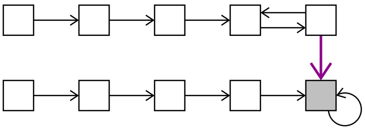
$$\nu(G) + 1 \leqslant \phi(G) \leqslant 2^{\tau^+(G)} \leqslant 2^{h(\nu(G))}.$$

→ Refine the bounds.

→ Show that $\phi(G) \leqslant 2^{h(\nu(G))}$ without using the theorem of Reed, Robertson, Seymour & Thomas, 1996 stating that $\forall G,\ \tau(G) \leqslant h(\nu(G))$.

→ What about limit cycles?

Last but not least: Structural robustness

Class ♯0 Perfect robustness	Class ♯1 Mean robustness
	
Class ♯2 **Weak robustness**	**Class ♯3** **Null robustness**
	

Noual, Regnault & S., 2012; Noual & S., 2017

The group

F. Bridoux

P. Guillon

T. Melliti

M. Noual

K. Perrot

D. Regnault

A. Richard

G. Theyssier